SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 2

MATH 241

17.4.2

Proof. The line is parametrized as $\{(-3, y)\}$. The mapping is $(x, y) \mapsto (u, v) = (x^2 - y^2, 2xy)$, so the image is parametrized as $\begin{cases} u = 9 - y^2 \\ v = -6y \end{cases}$. Cancel y, one gets $u = 9 - \frac{v^2}{36}$. □

17.4.38

Proof. Not analytic where the denominator vanishes, i.e., the zeroes of $z^2 - 6z + 25$. They are $3 \pm 4i$. At all points except for $3 \pm 4i$, one can find an open neighbourhood where the denominator does not vanish, hence the function is analytic. □

17.5.16

Proof.

$$\frac{\partial u}{\partial x} = 2x + ay, \frac{\partial v}{\partial y} = dx + 2y, \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow d = 2, a = 2$$

$$\frac{\partial v}{\partial x} = 2cx + dy, \frac{\partial u}{\partial y} = ax + 2by, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow 2c = -a, d = -2b$$

Hence $a = 2, b = -1, c = -1, d = 2$ will make the function analytic. □

17.5.19

Proof.

$$u = x^3 + 3xy^2 - x, v = y^3 + 3x^2y - y$$

$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 1, \frac{\partial v}{\partial y} = 3y^2 + 3x^2 - 1, \quad \text{so } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x} = 6xy, \frac{\partial u}{\partial y} = 6xy, \quad \text{so } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow xy = 0$$

So the Cauchy-Riemann equation is only satisfied on the points where $xy = 0$, i.e., the coordinate axes. This set does not contain any open subset, so it is not analytic at any point. □

17.5.21

Proof. We have seen $f'(x + iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$, hence if

$$f(z) = e^x \cos y + ie^x \sin y$$

then $f'(z) = e^x \cos y + ie^x \sin y$. □

17.5.28

Proof.

$$\frac{\partial u}{\partial x} = e^x(x \cos y - y \sin y) + e^x \cos y = \frac{\partial v}{\partial y}$$

Hence integrate to get $\frac{\partial v}{\partial y} = e^x x \sin y - \int e^x y \sin y dy + e^x \sin y + h(x)$. By integration by parts, we have

$$\int e^x y \sin y dy = -e^x y \cos y + e^x \sin y$$

Hence $v = e^x x \sin y + e^x y \cos y + h(x)$. Then $\frac{\partial v}{\partial x} = e^x(x \sin y + y \cos y) + e^x \sin y + h'(x)$. Compare with $\frac{\partial u}{\partial y} = e^x(-x \sin y - \sin y - y \cos y)$, we get we get $h'(x) = 0$, hence $h(x) = C$, and the conjugate harmonic funtion exists and is equal to $e^x x \sin y - \int e^x y \sin y dy + e^x \sin y + C$. □